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The Concept of Number Sequence in Graphical Representations for Secondary School Students

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Abstract: The aim of this work is to characterise the understanding that students in compulsory secondary education (14-16 years old) have of number sequences in graphical representations. The learning of numerical sequences is one of the first mathematical concepts to be developed in an infinite context. This study adopts the focus of semiotic representations as its theoretical framework. The participants consisted of 105 students and a qualitative methodology was used. The data collection instruments were a questionnaire and a semi-structured interview. The results allowed for three student profiles regarding number sequences in graphical representations to be identified. These profiles may facilitate a possible progression in the learning of number sequences for students in compulsory secondary education to be considered. Therefore, the results presented in this study can provide information about the learning hypotheses of mathematical tasks related to numerical sequences and can help in the design of such tasks.

Keywords: *Compulsory secondary education students, graphical representation, number sequences, progression in learning.*

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Introduction

Different studies have highlighted the importance of understanding the concept of sequences as a prerequisite for the understanding of other mathematical analysis concepts, such as understanding the concept of number series (Codes et al., 2013; Codes & González-Martín, 2017), in the concept of limits (Mamona, 1990; Roh, 2008; Sierpinska, 1990) or the introduction of the integral using Riemann sums (McDonald et al., 2000).

Moreover, studies examining the concepts of mathematical analysis have noted the relevant role that representations have in the teaching and learning of said concepts, limits (Pons et al., 2012) in the concept of function (Amaya, T., 2020; Tall & Vinner, 1981) in the concept of derivatives (Ariza & Llinares, 2009; García et al., 2011; Sánchez-Matamoros et al., 2006), in the concept of integrals (Boigues et al., 2010; González & Aldana, 2010; Orton, 1983; Aranda & Callejo, 2015), in the concept of number sequences (Biza et al., 2020; Cañadas, 2007; Montenegro et al., 2018; Przenioslo 2006; Rivera, 2013; Roh, 2008).

Furthermore, regarding the concept of number sequences, different studies have outlined the role of representations. Some papers have specifically been focused on determining whether representation formats influence how students solve tasks (Biza et al., 2020; Djasuli et al., 2017; Przenioslo, 2006; Roh, 2008). The results of these studies show not only the importance of different representations being present but also the transformations between them. More specifically, Cañadas (2007) highlights the importance of representations for the understanding of the presence of analytical (numerical, algebraic) and graphical (number line and cartesian plane) means of representation.

Regarding the concept of number sequence, in this paper, it is considered as the following:

"A sequence is an infinite set of numbers that are written in a specific order: $a_1, a_2, a_3..., a_n...$ The most fundamental point is that each member of the set has to be named with a natural subscript, with a_1 being the first and, a_2 the second, and in general, the nth term being a_n. We define the sequence as $\{a_n\}$, or simply a_n" (Stewart et al., 2007, p.178).

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In the same way, Stewart defines arithmetic progressions in the following way: "An arithmetic sequence or progression, AP, consists of the form a, a+d, a+2d,... a+nd... The number "a" is the first term and d is the common difference between two consecutive terms. The nth term is given by a_n= a+(n-1)d" (Stewart et al., 2007, p.181).

Lastly, Stewart defines a geometric progression as the following: "A geometric sequence or progression, GP, consists of the form a, ar, ar^2, ar^3,..., ar^n,... where a is the first element and "r ≠1" is the ratio of the progression. The nth term is given by $a_n=ar^{n-1}$. Unlike an arithmetic progression, where the difference between two consecutive terms is "d", in a geometric progression, the quotient between a_{n+1} and a_n is "r" for all "n" (Stewart et al., 2007, p.183).

There are many ways of expressing the terms in a number sequence: through the general term, by recurrence through a set of steps that allow a term to be obtained from the previous ones, and by extension by giving a series of consecutive ordered terms.

The aim of this study is to characterise the understanding that students in compulsory education have about the concept of number sequences through the coordination of two representations; graphical-linear and graphical-cartesian.

The theoretical framework regarding the semiotic representations in mathematics considered in this work (Duval, 2006) is outlined below. The qualitative methodology used in the study and the results obtained through the analysis of the data are also detailed. The paper finishes with the conclusions and a discussion of the study.

Literature Review

A common denominator in studies about students' understanding of mathematical concepts is the presence of representations. In this regard, many researchers note the presence and impacts of using representations: verbal, numerical, algebraic and graphic (Lesh et al., 1987), to gain an understanding of mathematical concepts (Altay et al., 2014; Biza et al., 2020; Callejo & Zapatera, 2014; El Mouhayar & Jurdak, 2016; Montenegro et al., 2018; Moyer-Packenham, 2005; Rivera & Becker, 2008; Warren & Cooper, 2008).

A representation in mathematics is defined by Goldin (2008) as a configuration that can substitute an entity in any form. These representations are important when solving tasks that involve mathematical concepts. Montenegro et al. (2018) consider that the multiple representations of a mathematical concept are one of the main difficulties that students find themselves struggling with.

Moreover, Duval (2006, 2017) notes that the change and coordination between different representations is a relevant factor for the understanding of mathematical concepts, and outlines that analysing the cognitive processes behind learning mathematics requires a change or orientation in the way tasks and problems are selected for students' learning. These representations, changes and coordination must be considered as cognitive variables. Therefore, it is considered that to reach a certain level of understanding, the change between representations and the coordination in the different representations are important (Duval, 2006, 2017).

The solving of a mathematical task is carried out in one representation but students must identify the same mathematical concept in different representations and use them. The representations used in the solving of a mathematical task always involve some transformation or conversion of semiotic representations.

Moreover, although individuals may use different semiotic representations, solving of a mathematical task requires that they only choose one. In other words, the solving of a mathematical task requires internal coordination between the representations without this coordination of the two different representations meaning two different concepts, and without there being any relation between them. Therefore, for Duval (2016), conceptual understanding arises from the coordination of various representations and not from only one of them.

This paper focuses on the graphical representation of the concept of number sequences. Two representations are considered: graphical-linear and graphical-cartesian. In the graph-linear representations, number sequences are represented as points on a number line (Figure 1); and in the graph-cartesian representations, number sequences are represented as points (n, an) on a cartesian plane (Figure 2).

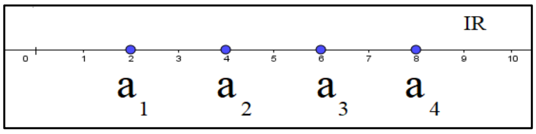


Figure 1. Graphical-linear representation

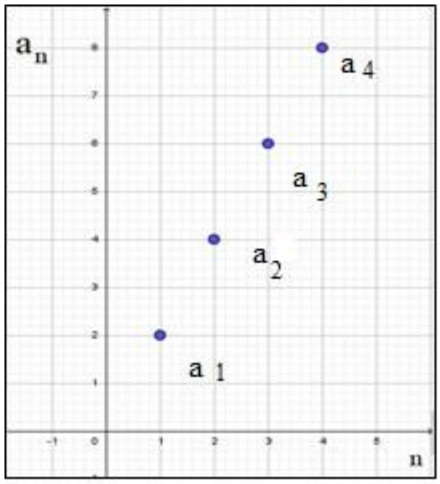


Figure 2. Graphical-cartesian representation

Methodology

Research Design

The methodology we are using is qualitative, using as a source of data two questionnaires of different nature, a first questionnaire of four tasks, which the students answered in a class session, and a second questionnaire designed for each student, based on the answers given to the first questionnaire, to deepen in those answers that either had not been argued, were not sufficiently justified or had not been answered.

In this section we present the research methodology, which we have divided into three parts. In the first part we present the participants in the study, in the second part we present the data collection instruments used to carry out our research, and finally, we present the design of the analysis procedure, which has guided us in achieving our objectives for this study.

Participants

In this study, 105 students participated. They were in their final years of compulsory secondary education (14-16 years old) in a center offering this educational stage in a Spanish city. The students studied the didactic unit about number sequences in accordance with their official syllabus: the concept of sequence, finding regularities in sequences with integer and rational terms, arithmetic and geometric progressions (Official State Bulletin, 2015).

Data Collection Instrument

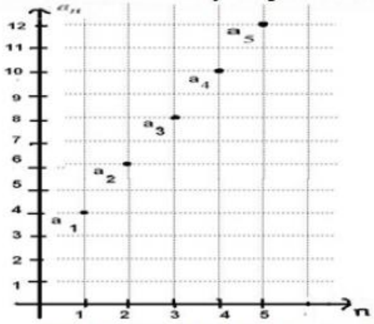
In this study, the data collection instrument consisted of two tasks that made up a more extensive questionnaire, which was based on a literature review of research on number sequences, Figures 3 and 4.

The students responded to the questionnaire during class time. Once an analysis of the students' responses and reasoning had been done, a semi-structured interview was carried out with each participant. The objective of the interview was to obtain further details about the responses that had not been explained or that brought about doubts in their interpretation. In the interview, the students had their written answers to the first questionnaire.

Task 1 (Figure 3) presented a number sequence in a graphical-cartesian representation and, in order to solve it, students were required, in section a), to identify whether the points of the cartesian plane (n, a_n) corresponded with the nth term of the given sequence. In this way, and using a conversion to a numerical representation, students needed to identify that $a_2=6$ was the equivalent of the point (2,6). Likewise, an analysis needed to be done to obtain a_4 and a_6.

Tarea 1

Sea la sucesión cuya representación gráfica sobre los ejes cartesianos es la siguiente:



a) ¿Cuál es el término segundo, es decir, a_2? , ¿Y el cuarto a_4? ¿Y el sexto a_6?
b) ¿Hay algún término que valga 16? ¿Y 13? Razona las respuestas.
c) ¿Es creciente o decreciente? Justifica la respuesta.
d) ¿Qué pasa con a_n cuando n se hace cada vez mayor?

Translation

Task 1

Let there be the number sequence whose graphical representation on the cartesian axes is as follows

a) What is the second term, i.e. a_2? and the fourth term a_4 ? and the sixth term a_6 ?

b) Is there any term which is 16? and 13? Reason the answers

c) Is it increasing or decreasing? Justify the answer

d) What happens to a_n when n gets bigger and bigger?

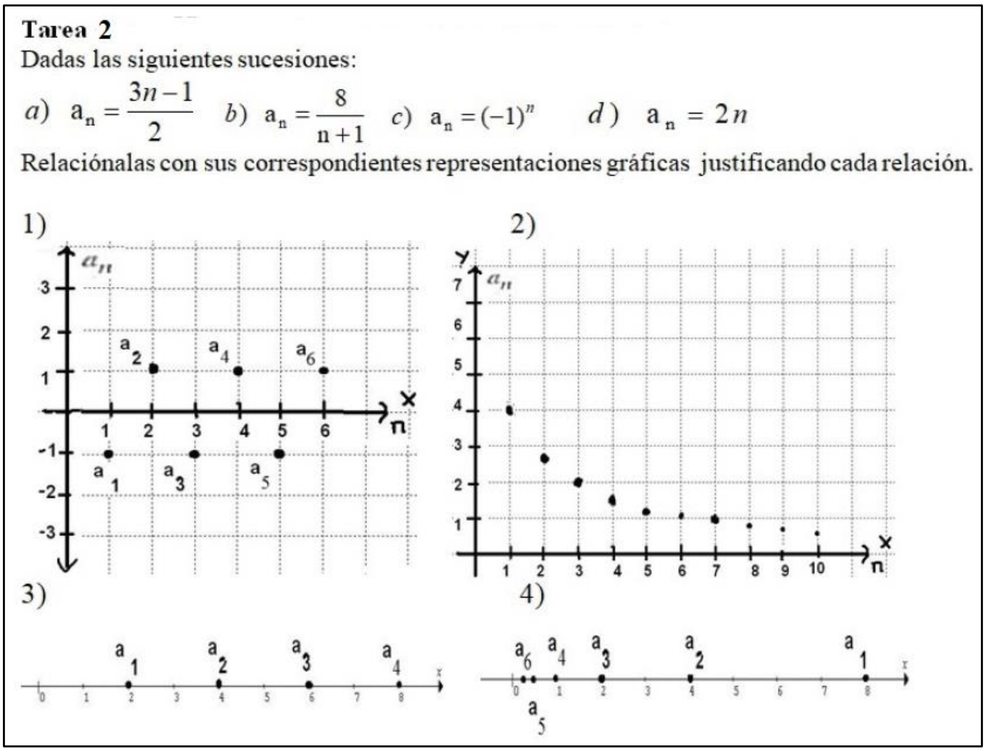
Figure 3. Task 1 of the questionnaire

In order to respond to section b), students needed to follow the same method as for section a) but also needed to explicitly mention that the value of the abscissa had to be a natural number for it to be a term in the sequence and that this value also indicated which term of the sequence it was.

For section c), students needed to analyse the data to compare consecutive terms. This analysis could be done directly on the graph or through a conversion to the numerical representation. They also needed to interpret the data to infer the behaviour of terms that were not specified in the graphical-cartesian representation.

For section d), the students needed to follow the same method as section c) and interpret the data again. The expected answers could have been, among others, their strictly increasing monotonicity or divergence.

In task 2 (Figure 4), sequences in three representations were presented: algebraic expressions, and graphical-linear and graphical-cartesian representations. Students were asked to relate the algebraic expression with one of the graphical-linear or graphical-cartesian representations. Not all the algebraic expressions of the sequences provided could be paired with one of the graphical representations; specifically, the number sequence in section a). Similarly, not all the graphical representations could be paired with an algebraic representation; in this case, specifically, graphical-linear representation 4.



Translation

Task 2

Given the following numbers sequences:

Relate them to their corresponding graphical representations, justifying each relation.

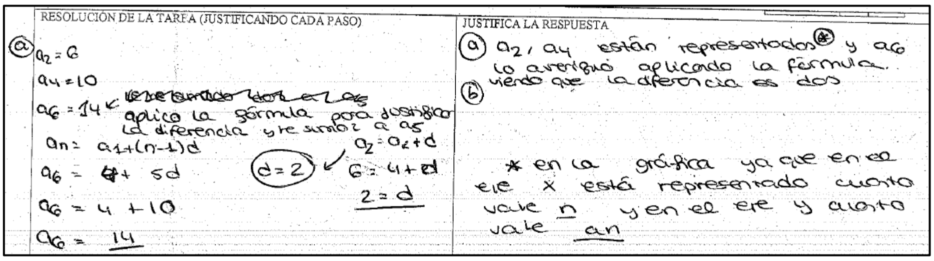
Figure 4. Task 2 of the questionnaire

To solve the task, students could find some concrete terms in the algebraic representation, find the equivalence in the graphical representation (linear or cartesian) and make a subsequent comparison. If one of the values did not correspond with the comparison, the students needed to discard the corresponding representation. If all values matched, the students needed to interpret that the same was true for all the values that were not explicitly listed.

Analysis Procedure

The analysis was focused on identifying the use of graphical-linear and graphical-cartesian representations, as well as the transformations and conversions that could be identified in the students' answers. The data from the two data sources (the written questionnaire and the subsequent semi-structured interview which was personalised for each student according to the answers given to the written questionnaire) were considered together. We will now show, by means of an example, how we carried out the analysis procedure using the two data sources.

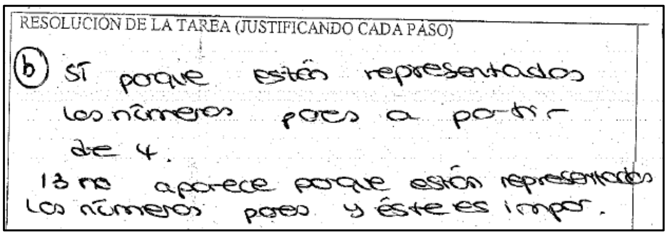
Regarding the graphical-cartesian representation, in the first section of task 1 given in this representation, student e12 had to carry out a conversion to a numerical representation to be able to respond to the section, in which they were asked for the value of certain terms of the sequence a_2, a_4 and a_6. This student correctly analysed the graphical-cartesian representation as they correctly assigned meaning to the role of each of the axes in it (the x-axis was the location of the term of the sequence and the y-axis was the value of the term of the sequence (n, an)). This can be seen in Figure 5, as interpreting the point (1,4) and the point (2,6) gives an answer of $a_2=a_1+d$. 6=4+d, then d=2, $a_4=10$, $a_6=14$, carrying out the conversion from the graphical-cartesian representation to the algebraic representation and then to the numerical representation correctly.



Translation:	Justify the answer
Resolution of the task (justifying each step) I apply the formula to justify the difference and add 2 to a_5.	a) a_2, a_4 are represented in the graph and that in the x-axis is represented when it is n and in the y-axis when it is an and a_6 I find out by applying the formula seeing that the difference is two.

Figure 5. Answer of student e12 to item a) of the questionnaire task 1

In response to section b) from task 1, this student responded that 16 was a term in the sequence but that 13 was not as the sequence terms were all even (Figure 6).



Translation:
b) yes because all the even numbers from 4 onwards are represented 13 Does not appear because the even numbers are represented and this one is odd.

Figure 6. Answer of student e12 to item b) of the questionnaire task 1

Later, in the semi-structured interview, we asked the student to clarify their response. They told us that, when carrying out a conversion to an algebraic representation, the subscript n needed to be a natural number. The joint consideration of the questionnaire and the interview guarantees the reliability of the inference made (triangulation of data):

Question: In section b from task 1, does the simple fact of being odd ensure that there is no term? Explain.

e12: $a_n= 2 +2n \Rightarrow a_7 = 2 + 2n = 16$,

$13 = 2 + 2n \Rightarrow 11 = 2n \Rightarrow n= 11/2$ $n = 5{,}5 \Rightarrow$ no, because n has to be a natural number.

Findings / Results

The results from the analysis of the students' responses to the questionnaire and the semi-structured interview allowed us to characterise the different uses of the graphical representations, which are shown in the following table (Table 1).

Table 1. Number of students in each of the graphical representations

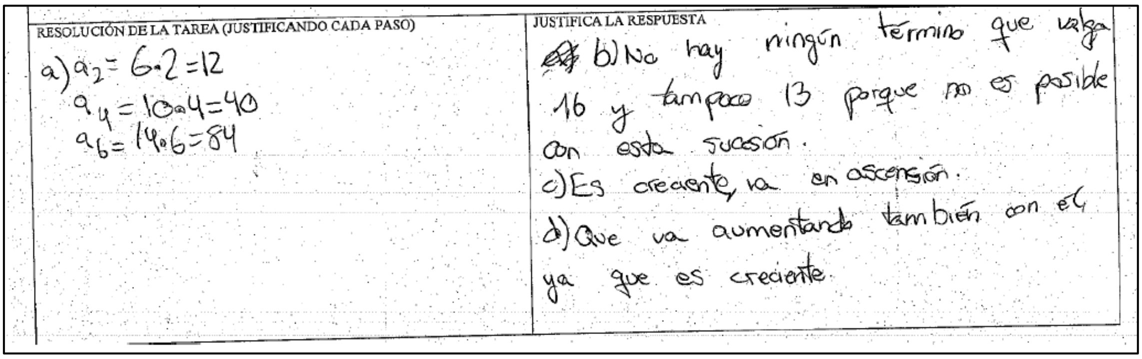
Representation	Number of students
Incorrect use of the graphical representation	12
Correct use of only the graphical-linear representation	15
Correct use of the graphical representations	78

We will now describe each of them.

Incorrect use of the graphical representation

The students from this group were characterised by an incorrect use of both the graphical-linear representation and the graphical-cartesian representation when solving the tasks.

An example of this error can be seen in student e31. When solving task 1, this student did not correctly analyse the graphical-cartesian representation since, when correctly converted to a numerical representation, the value of a_2 was 6. However, this student considered that a_2 equalled 6•2=12, and acted accordingly with a4 and a6 (Figure 7).



RESOLUCIÓN DE LA TAREA (JUSTIFICANDO CADA PASO)	JUSTIFICA LA RESPUESTA
a) $a_2 = 6 \cdot 2 = 12$ $a_4 = 10 \cdot 4 = 40$ $a_6 = 14 \cdot 6 = 84$	b) No hay ningún término que valga 16 y tampoco 13 porque no es posible con esta sucesión. c) Es creciente, va en ascensión. d) Que va aumentando también con el, ya que es creciente.

Translation:

b) There is no term worth 16 and no term worth 13 with this number sequence.

c) It is increasing, it is ascending

d) It is also increasing with it, since it is increasing.

Figure 7. Answer of student e31 to task 1 of the questionnaire

We confirmed this in the semi-structured interview:

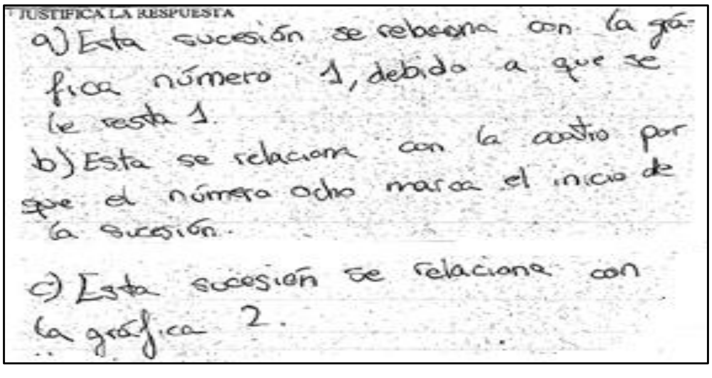
Question: For task 1, can you explain how you calculated the sequence terms using the graphic?

e31: As it was the cartesian plane, what I did was place the cartesian product of the numbers below and above and what I got was the value of each term.

This student responded to sections b), c) and d) carrying out the same conversion of the graphical-cartesian representation to the numerical representation as in the previous section.

In task 2, the student incorrectly analysed the graphical-cartesian representation. In all sections of this task, the students needed to identify the same sequence in two different representations by carrying out a conversion. One of the representations was algebraic and a conversion needed to be done in section d) with the graphical-linear representation in section 3 and in section c) and b) with the graphical-cartesian in sections 1 and 2, respectively.

Moreover, student e31 again carried out incorrect conversions between the algebraic and graphical-cartesian representations as they linked section c) with 2) and similarly carried out an incorrect conversion between the graphical-linear representation and the algebraic representation, due to an incorrect analysis of the graphical-linear representation, which became apparent when linking section b) with 4), as shown in Figure 8.



JUSTIFICA LA RESPUESTA

a) Esta sucesión se relaciona con la gráfica número 1, debido a que se le resta 1.

b) Esta se relaciona con la cuatro por que el número ocho marca el inicio de la sucesión.

c) Esta sucesión se relaciona con la gráfica 2.

Translation:

a) This number sequence is related to graph number1, because 1 is subtracted from it

b) This number sequence is related to four because the number eight marks the beginning of the number sequence.

c) This number sequence is related to graph number 2

Figure 8. Answer of student e31 to task 2 of the questionnaire

In the semi-structured interview, we asked the student about this and they confirmed that their errors came from incorrectly analyzing thegraphical-cartesian and graphical-linear representations.

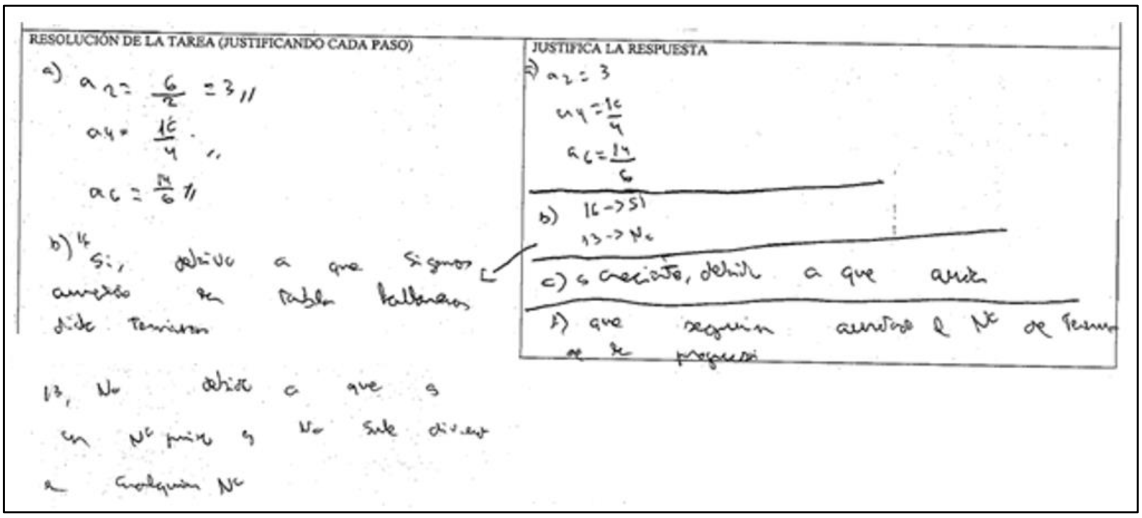
Question: For task 2, can you explain sections c) and b)?

e31: In section c, it is 2 because, with it being negative, it is worth less and less and has a downward trend. In section b, the first value is 8 and, therefore, I found the 8 in the line.

Correct use of only the graphical-linear representation

The students from this group are characterised by the correct use of the graphical-linear representation when solving the task using a conversion (converting to a numerical or algebraic representation) when necessary. However, the students were not correct when solving the task that required them to use the graphical-cartesian representation.

An example of this type of student can be seen in student e17. When solving task 1, this student did not correctly analyse the graphical-cartesian representation since, when correctly converted to a numerical representation, the value of a_2 was 6. However, the student considered that the second term a_2 was 6: 2=3, and acted accordingly with a_4 and a_6 (Figure 9).



Translation	Justify the answer
Resolution of the task (justifying each step)	b) 16 yes
b) 16 yes because if we increase n in the table we arrive at the term	13 No
13 no because it is a prime number and does not divide by any number.	c) It is increasing, because it increases d) That we keep increasing the number of the term of the number sequence.

Figure 9. Answer of student e17 to task 1 of the questionnaire

We confirmed this in the semi-structured interview:

Question: For task 1 can you explain how you calculated the sequence terms using the graphic?

e17: I calculated them using the two values from the line, the top one divided by the bottom one and I got the value of the final number, I think.

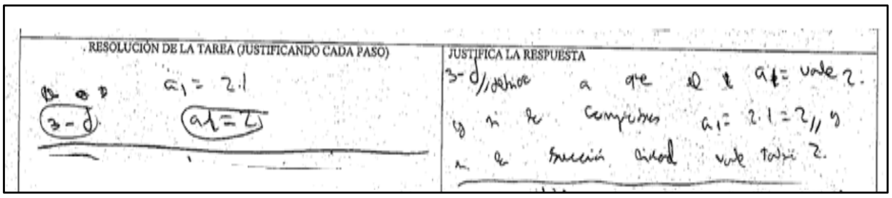
Question: Can you further explain section b)?

e17: 16 is indeed a value because you can divide it and get the number but 13 cannot do the same because as it is a prime number, it cannot be divided by another number to make 13.

This student incorrectly responded to section b) as they considered divisibility, which was not related to the question. The student solved sections c) and d) with the same analysis errors in the graphical-cartesian representation as in the previous section.

This same student, for all sections from task 2, needed to identify the same sequence in two different representations by carrying out a conversion. One of the representations was algebraic and the conversion needed to be carried out in section d) with the graphical-linear representation in section 3, and in section c) and b) with the graphical-cartesian in sections 1 and 2, respectively.

When the task needed a graphical-linear representation for its solution, student e17 used it correctly. This fact can be seen when, in answering, the student related section d in the algebraic representation given by the analytical expression $a_n = 2n$ with the graphical-linear representation of graph 3. The student wrote: "3-d, because a1 is 2 and if we check $a_1 = 2 \cdot 1 = 2$ and the linear function is also 2", (see Figure 10).



Translation:

3-d) Because a_1 is 2 and if you check a_2 = 2-1 = 2, and the linear function is also 2.

Figure 10. Answer of student e31 to task 2 of the questionnaire

Therefore, we can consider that this student correctly converted the algebraic representation to a numerical representation and subsequently to a graphical-linear representation, as they confirmed in the semi-structured written interview.

Question: So, what type of graphics do you think are the easiest to use for your interpretation?

e17: Those that are on one line [referring to the graphical-linear representations] as you can see the values faster and they are much better represented. I have more problems with the ones that are two numbers above and below [referring to the graphical-cartesian representations].

In summary, when using the graphical-cartesian representation, the student had many difficulties in their interpretation, but when they used the graphical-linear representation, they correctly did so, carrying out correct conversions between the algebraic representation and the graphical-linear representation.

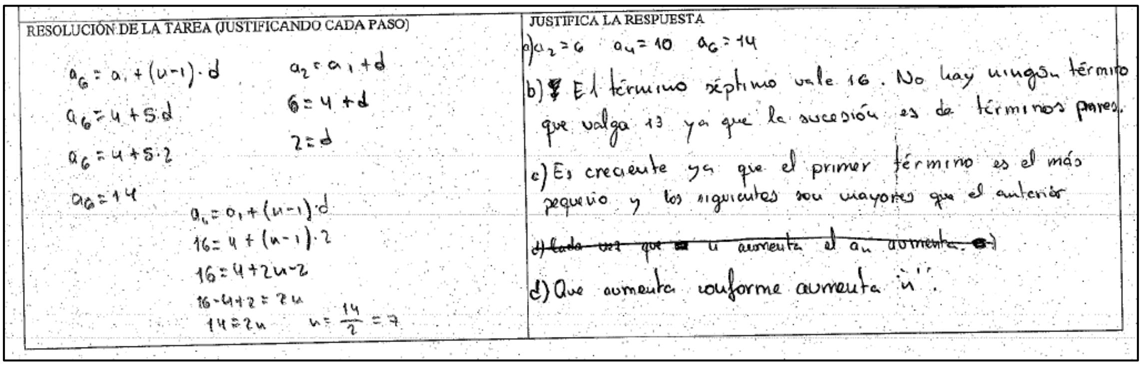
Correct Use of Graphical Representation

The students of this group are characterised by a correct use of both the graphical-linear and graphical-cartesian representation when solving the tasks using a conversion (converting to a numerical or algebraic representation), when necessary.

An example of this can be found with student e92 who correctly used the graphical-cartesian representation. This can be seen when analysing the point (1,4), which led the student, through a conversion to the numerical representation, to the fact that a_1=4. The same happens with point (2,6). This allowed the student to find the value of the difference of the arithmetic progression given in the task. Subsequently, the student used the formula that corresponded to the general term of an arithmetic progression to answer the different sections of the task. They obtained the sixth term correctly and answered that there was a term with a value of 16 and there was no term with a value of 13 (Figure 11).

For section c), the student analysed the data to compare consecutive terms and interpreted the data to infer the behaviour of the terms that were not specified in the graphical-cartesian representation.

For section d), the student continued in a similar way to the previous section, inferring the behaviour of the whole number sequence.



RESOLUCIÓN DE LA TAREA (JUSTIFICANDO CADA PASO)

$a_6 = a_1 + (n-1)\cdot d$ $a_2 = a_1 + d$
$a_6 = 4 + 5\cdot d$ $6 = 4 + d$
$a_6 = 4 + 5\cdot 2$ $2 = d$
$a_6 = 14$
$a_n = a_1 + (n-1)\cdot d$
$16 = 4 + (n-1)\cdot 2$
$16 = 4 + 2n - 2$
$16 - 4 + 2 = 2n$
$14 = 2n$ $n = \frac{14}{2} = 7$

JUSTIFICA LA RESPUESTA

a) $a_2 = 6$ $a_4 = 10$ $a_6 = 14$
b) El término séptimo vale 16. No hay ningún término que valga 13 ya que la sucesión es de términos pares.
c) Es creciente ya que el primer término es el más pequeño y los siguientes son mayores que el anterior.
~~d) cada vez que n aumenta el a_n aumenta~~
d) Que aumenta conforme aumenta "n".

Translation:

a) a_2=6 a_4=10 a_6=14

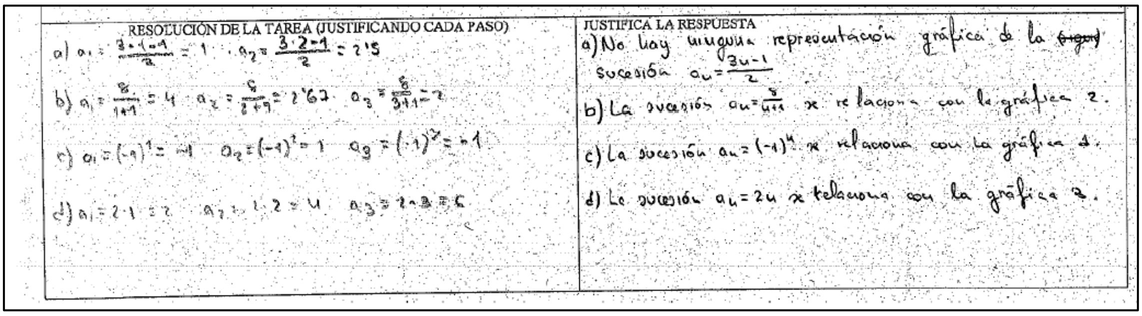
b) The seventh term is worth 16. There is no term that is worth 13 since the number sequence is of even terms.

c) It is increasing, the first term is the smallest and the following terms are larger than the previous one.

d) Increasing as "n" increases.

Figure 11. Answer of student e92 to task 1 of the questionnaire

In task 2, the student used the general term from the four task sections in the algebraic representation and carried out a conversion to a numerical representation in order to find the three first terms. The student also analysed the graphical representation (linear or cartesian) and made a subsequent comparison. For this student, the number sequences b) and c) coincided with their corresponding graphical-cartesian representations and section d) with its corresponding graphical-linear representation. Likewise, they did not pair the number sequence a) with any graphical representation nor the graphical-linear representation 4 with any algebraic representation (Figure 12).



RESOLUCIÓN DE LA TAREA (JUSTIFICANDO CADA PASO)

a) $a_1 = \frac{3\cdot 1-1}{2} = 1$; $a_2 = \frac{3\cdot 2-1}{2} = 2'5$
b) $a_1 = \frac{8}{1+1} = 4$ $a_2 = \frac{8}{2+1} = 2'67$ $a_3 = \frac{8}{3+1} = 2$
c) $a_1 = (-1)^1 = -1$ $a_2 = (-1)^2 = 1$ $a_3 = (-1)^3 = -1$
d) $a_1 = 2\cdot 1 = 2$ $a_2 = 2\cdot 2 = 4$ $a_3 = 2\cdot 3 = 6$

JUSTIFICA LA RESPUESTA

a) No hay ninguna representación gráfica de la sucesión $a_n = \frac{3n-1}{2}$
b) La sucesión $a_n = \frac{8}{n+1}$ se relaciona con la gráfica 2.
c) La sucesión $a_n = (-1)^n$ se relaciona con la gráfica 1.
d) La sucesión $a_n = 2n$ se relaciona con la gráfica 3.

Translation:

a) There is no graphical representation of this number sequence
b) This number sequence is related to the graph 2
c) This number sequence is related to the graph 1
d) This number sequence is related to the graph 3

Figure 12. Answer of student e92 to task 2 of the questionnaire

In summary, student e92 correctly used the different representations, showing conversions between the graphical (linear and cartesian), numerical and algebraic representations.

Discussion

The use of different representations has been considered as a key element for characterising the understanding of mathematical concepts. This work examines the consideration and differentiation of two graphical representations that are related through conversion; graphical-linear and graphical-cartesian representations. Both representation types were characterised using two processes: reading and interpretation. On the one hand, reading is a process that we can consider punctual as it is linked to concrete points (particular terms from a number sequence, either to a single point or as a finite set of terms in the sequence). On the other hand, interpretation is an overall process in nature, that is, it is linked with the complete number sequence with its infinite terms. Our results show that the cartesian graphical representation register presents greater difficulty than the linear graph because the former requires a double reading, one to coordinate the position with the value of the term of the numerical sequence, while the latter requires only one reading to coordinate the position with the value of the term of the numerical sequence.

The results of this paper are in line with those of other studies that note the presence of representations and the relevance of the conversions between them for understanding the concept of number sequences (Biza et al., 2020; Cañadas, 2007; Djasuli et al., 2017; Przenioslo, 2006; Roh, 2008). Moreover, our study is in line with the results obtained by Biza et al. (2020). The most used strategy among the students in this study to solve the task involving different representations was, as can be seen in the results, the conversion to a numerical representation.

Moreover, our results agree with Duval (2006, 2017) who considers that for a concept, each representation has some, but not all, of its own characteristics associated with it. In this way, conceptual understanding arises from the coordination of several different representations. Žakelj and Klančar (2022) show the importance of the use of representation registers, since they develop the cognition of mathematical concepts. The concept of number sequences in a graphical representation has different associated characteristics according to whether it is a graphical-linear or graphical-cartesian representation. In the former, there is no need to coordinate the position and the value of the term, and in the latter coordination is needed in all situations. We cannot consider that the student has conceptually understood the graphical representations of number sequences until they have carried out the coordination process. It is noteworthy that this work has shown that not all mathematical concepts make use of representations in the same way, as can be seen in the different uses of graphical-linear and graphical-cartesian representations.

Conclusions

The results obtained in this study allow for the consideration that differentiating between graphical-linear and graphical-cartesian representations can help to characterise progression in learning the concept of number sequences in a graphical representation. All students who correctly used the graphical-cartesian representation, also correctly used the graphical-linear representation, but not vice versa. Due to this, it can be concluded that learning number sequences using graphical representations takes place by first correctly using the graphic-linear representation and then moving on to correctly using the graphical representation in its two forms (graphical-linear and graphical-cartesian).

This work demonstrates that the two graphical representations of a sequence are not the same; a graphical-linear representation is shown in one dimension (real line) and requires different analysis skills to those needed for a graphical-cartesian representation, which is shown in two dimensions (cartesian plane). Due to this, conversions between the two types of representations are not always instinctive. To be better able to carry out conversions between the graphical representations, it is recommended that a conversion to a numerical representation is done.

Recommendations

Learning number sequences is a challenge for students in secondary education as it is one of the first mathematical concepts that they do not develop in a finite context, but rather, an infinite one. More specifically, this particular discrete infinite context (based on natural numbers, which is an infinite countable set) is the transition to mathematical analysis concepts in a non-discrete infinite context (continuous, non-countable set) that is based on real numbers. This study has focused on this concept, putting an emphasis on graphical representations. These types of studies are vital when designing teaching plans at these educational levels that aim to help students understand particular concepts.

These ideas, as outlined in this study, may be useful for teaching when the teacher develops prospective learning plans (Simon, 1995). For the development of these plans, this work may provide information regarding the learning hypotheses of mathematical tasks relating to numerical sequences and can help in the design of such tasks. More specifically, and as we have shown, this study may be useful in the learning progression of the student regarding graphical representations of number sequences, moving from the correct use of graphical-linear representations to the correct use of both graphical-linear and graphical-cartesian representations.

Limitations

Given that this work has been carried out with compulsory secondary school students (14-16 years old), a future line of research is to carry out the research with students at pre-university or university levels and to investigate the use of the graphic representation register by these students.

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Authorship Contribution Statement

Bajo-Benito: Conceptualization, methodology, analysis, results, discussion, conclusions, writing—original draft preparation and writing—review and editing. Gavilán-Izquierdo, Conceptualization, methodology, analysis, results, discussion, conclusions, writing—original draft preparation and writing—review and editing. Sánchez-Matamoros García: Conceptualization, methodology, analysis, results, discussion, conclusions, writing—original draft preparation and writing—review and editing.

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