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How Students Generate Patterns in Learning Algebra? A Focus on Functional Thinking in Secondary School Students

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Abstract: This research aims to describe secondary school students' functional thinking in generating patterns in learning algebra, particularly in solving mathematical word problems. In addressing this aim, a phenomenological approach was conducted to investigate the meaning of functional relationships provided by students. The data were collected from 39 ninth graders (13-14 years old) through a written test about generating patterns in linear functions. The following steps were conducting interviews with ten representative students to get detailed information about their answers to the written test. All students' responses were then analyzed using the thematic analysis software ATLAS.ti. The findings illustrate that students employed two types of approaches in solving the problem: recursive patterns and correspondence. Students favored the recursive patterns approach in identifying the pattern. They provided arithmetic computation by counting term-to-term but could not represent generalities with algebraic symbols. Meanwhile, students evidenced for correspondence managed to observe the relation between two variables and create the symbolic representation to express the generality. The study concludes that these differences exist due to their focus on identifying patterns: the recursive pattern students tend to see the changes in one variable, whereas the correspondence ones relate to the corresponding pair of variables.

Keywords: *Functional relationships, functional thinking, generalization, learning algebra.*

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Introduction

The emergence of early algebra was discussed by literature since learning arithmetic and algebra separately left the students incapable of expanding their thinking from concrete to abstract (Carraher & Schliemann, 2007; Kieran et al., 2016). To address the issue, different studies suggested that arithmetic-algebra transition should be embedded in elementary mathematics (Carraher et al., 2006). Early algebra is essential because it fosters the identification of mathematical relationships rather than isolated arithmetic (Cai & Knuth, 2005; Malisani & Spagnolo, 2009). Generally speaking, Kaput (2008) proposed two essential aspects for algebraic thinking in early grades: generalization and symbolization of mathematical relationships. Therefore, students attend to examining relationships between quantities and express their regularities. Specifically, this study emphasizes on functional thinking, in which function is the prime mathematics topic.

Functional thinking become apparent because students encounter problems that required this ability to solve. Students discover number patterns in preschool and elementary school (Blanton & Kaput, 2005; National Council of Teachers of Mathematics [NCTM], 2000), generalize geometric patterns and learn functions in secondary school, which are fundamental for learning calculus in higher education (NCTM, 2000). Taking these topics appropriately, students must develop an understanding of the functional relationship between co-varying quantities and be proficient in functional thinking (Lichti & Roth, 2018).

Different studies have reported how students identify, generalize, and express functional relationships (Pinto & Cañadas, 2021; Ramírez et al., 2022; Wilkie, 2016). In common, elementary graders failed to recognize the pattern within a sequence of values (Wilkie & Clarke, 2016), and they exhaustively counted the objects from the first term to the requested

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one (Radford, 2006, 2010). A similar result was also experienced by secondary students (7-9 grade), who tended to count the numbers in consecutive numerical steps (El Mouhayar, 2018).

Furthermore, there are studies assessing how students solve functional thinking problems that are significantly centered on the meaning of variable notations (Ayala-Altamirano & Molina, 2020; El Mouhayar, 2018; Pinto et al., 2022). The frequent representation use by students to express regularity was natural language, and few answered with mathematical symbols (Wilkie, 2016). Additionally, learners from elementary grades still used variable notations to replace labels or objects, determined the values of variables based on alphabetical order (e.g., $a = 1, b = 2$) (Blanton et al., 2017; Lucariello et al., 2014; Malisani & Spagnolo, 2009), and operated the same letters to two different variables (Ayala-Altamirano & Molina, 2020). Indeed, students' conception of variables has become the outline for studies for years, where they ought to understand the versatile role of variables (Usiskin, 1988; Wagner & Parker, 1993). In functional thinking, students need to understand variables from a broader perspective, that is, to represent co-varying quantities instead of solely the unknown (Doorman et al., 2012).

Nonetheless, some topics have not been addressed in depth in recent studies. Most literature contributed to functional thinking in elementary grades. Indeed, early algebra has been implemented in primary-grade curricula in developed countries such as the United States, Spain, and Japan (NCTM, 2000; Pinto & Cañadas, 2021; Watanabe, 2011). However, early algebra is not yet included in primary school mathematics in Indonesia, resulting in difficulties to learn algebra when learners reach secondary school. Additionally, since geometric patterns are ubiquitous in portraying elementary grades' functional thinking, focus on functional relationships between variables in word problems has not yet been explored deeply in recent literatures. Word problems, *mathematical word problems*, are verbal descriptions of problems wherein the mathematical concepts are designated to be used as the solution for the problem (Verschaffel et al., 2020). Thus, the mathematical word problem focused on in this study will be solved using the concept of linear functions.

Research Objective

Therefore, and considering the above reasons, this study's general objective is to describe secondary school students' functional thinking when generating patterns in mathematical word problems. Accordingly, two research questions are addressed as follows:

1. How are the functional relationships evidenced by secondary school students in identifying patterns from specific instances in mathematical word problems?
2. How is the student's ability to symbolize the generalization according to their functional relationships approach?

Literature Review

Functions and Functional Thinking

The concept of function has evolved across centuries from different perspectives, beginning with the curve movements in analytical geometry, analytical expressions showing relationships within variables, and arbitrary pairing between elements in two sets (Kleiner, 1989). The modern concept of function defined by Dirichlet-Bourbaki (in Vinner & Dreyfus, 1989) is a rule that assigns each element in a non-empty set (domain) to exactly one element in another non-empty set (codomain). Moreover, this study focuses on linear functions. This type of function, even the most uncomplicated linear functional relationships, becomes challenging for students of any age (Nistal et al., 2012). Typically, secondary graders exhibit a context-specific knowledge of linear functions than becoming connected with different representations (Wilkie & Ayalon, 2018).

As the prerequisite for calculus, Algebra entails the development of the concept of a function, exceptionally functional thinking (Lichti & Roth, 2018). Functional thinking is an ability that focuses on the relationship between two (or more) varying quantities (variables) in mathematics (Smith, 2008). In functional thinking, the function is employed as the basis for a representational system generated by students to express a generalization from specific relationships to that relationship across instances (Günster & Weigand, 2020; Lichti & Roth, 2018). Functional thinking involves the heart of children's algebraic reasoning: generalization and symbolization (Kaput, 2008; Smith, 2008). Generalization is the process of pattern recognition from identified objects (Septiani et al., 2018), while symbolization is the process of employing a symbolic system to express generalization (Kaput, 2008).

Students' functional thinking can be evidenced by how they identify the relationships between variables: *recursive patterns, covariation,* and *correspondence*. The students can create a record to understand locally how each value in a variable is assigned to a unique value in another variable (*recursive patterns*). Students might focus on how the dependent variable alters when the independent variable is varied (*covariation*) until they identify the correlation between variables (*correspondence*) (Confrey & Smith, 1994; Doorman et al., 2012; Smith, 2008; Wilkie, 2014). To give an illustration of how students employ these functional relationship approaches, Figure 1 portrays an example of possible students' responses when they are engaged with tasks involving the generalization of relationships between quantities drawn from particular instances.

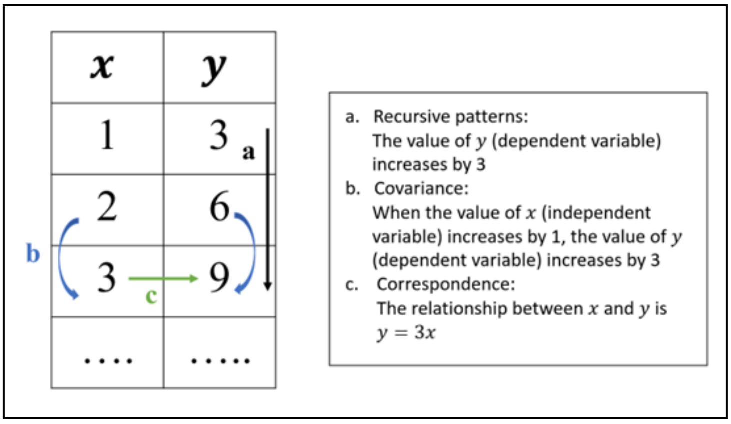


Figure. 1. Example of functional relationships representation between variables

Confrey and Smith (1995) stated that displaying functions with tables (Figure 1) might induce students to coordinate the two variables and describe the functional relationship with covariation. However, even with tables of values, students did not necessarily attend to covariational values but relied on term-to-term changes (recursive) in each variable (Carraher & Schliemann, 2007). Different studies proved that elementary graders failed to recognize the pattern within a sequence of values (Wilkie & Clarke, 2016). They exhaustively counted the objects from the first term to the requested one (Radford, 2010; Radford, 2006). A similar result was also experienced by secondary students (7-9 grade), who tended to count the numbers in consecutive numerical steps (El Mouhayar, 2018).

Furthermore, there are different approaches to engaging students in functional relationship problems and nurturing their functional thinking. However, there were two types of tasks that were widely used in prior studies: (1) making the functional situation a part of the problem statement, e.g., *Carlos wants to sell shirts where he can earn 3 euros for each sold shirt* (Pinto et al., 2022) and (2) making the functional situation listed in a table (Figure 1). Unfortunately, the first case might preclude the students from discovering the pattern. Meanwhile, the second case demands pattern recognition but may be cut off from the problem's context (Smith, 2008). Thus, this study will elaborate on the type of problems assigned to the students from previous studies. Regarding the context of this study, the problem from (1) and (2) can be described as (3) *Carlos earns 3 euros if he sells a shirt and earns 6 euros if he sells two shirts. How many euros does he earn if he sold n shirt?*

Methodology

Research Design

A phenomenological approach (Moustakas, 1994) was employed in this study since it aimed to interpret and describe the students' way of generalizing patterns and creating the formula for expressing functional relationships between variables through exploring their experiences in obtaining that solution.

Participant and Data Collection

Thirty-nine 9 grade students (13-14 years old, both male and female) from one of the secondary schools in West Java, Indonesia, acted as the participants in this study. Initially, this study planned to have eight-grade students as the research participants since the topic of function was introduced at this level. However, by the time the data for this study was collected, eighth graders had not yet learned about functions. After thoroughly discussing with co-researchers and the mathematics teacher, it was decided to opt for 9 graders. Additionally, all 39 students voluntarily participated in this study since they all agreed to do the test and interview.

The researchers acted as the main instrument in this study and were directly involved in all data collection processes. The supporting instruments included a designated test about functional thinking tasks in word problems, student interview guidelines, and documentation (audio recordings). The data collection was begun by administering an individual test to all 39 students. The test involved three questions, which aimed as the entry point to capture the students' functional thinking in solving word problems.

Problems presented to students (Appendix) were selected according to three criteria: (a) the type of function used in each question, (b) the kind of context used in each problem that possesses the possibility of students' generalization according to the functional relationship's characteristics (*recursive patterns, covariation,* and *correspondence*), and (c) the level of difficulties, that was from less to more difficult. Moreover, the test consisted of three problems. Each problem had a different linear function rule ($y = ax, y = ax + b$, and $y = ax - b$), respectively. Three experts in mathematics education theoretically validated these functional thinking tasks to ensure their appropriateness to secondary school students and the functional relationship's characteristics. Additionally, the test lasted for about thirty minutes. In this test, students were not allowed to use calculators nor work with their peers.

Furthermore, the interview proceeded after the students' test answers were obtained. In this study, ten students were selected for the additional individual interviews based on a preliminary selection by observing all students' answers to the written test. It was done to gather more detailed information on the functional relationships generated by students in each problem. In this way, the researcher conducted the interviews and audio-recorded for approximately 10-15 minutes per student. During the semi-structured interviews, each student's written answers were presented, and they were encouraged to explain their reasoning. A guideline for carrying out the interview was made of general starting questions for investigating students' functional thinking and could be adjusted as necessary to allow flexibility during the interview. The general questions included: Have you ever solved problems like this? Do you understand this problem? Can you explain the information and question stated in this problem? How do you solve this problem? Is there any difficulty you found when solving this problem?

Analyzing of Data

A qualitative data analysis software, ATLAS.ti, was used in this study's whole data analysis process. The tool has been conducted in qualitative educational research, including phenomenological studies (Lukman et al., 2021; Woods et al., 2016). Moreover, the following procedures were used for the data analysis process in this study are as follows:

(1) Becoming familiar with the data

The initial phase was done by repeatedly reading students' written answers and interview transcripts. In addition, the researchers also constantly listened to the audio recordings to ensure that the interview transcripts were appropriate to the actual interview results. Additionally, familiarizing ourselves with the data allowed researchers to predict the characteristics of every piece of information and enabled researchers to make the initial codes. The researchers began to use ATLAS.ti when the students' written answers and interview transcripts were imported into the software.

(2) Generating initial codes

Every statement in the data that supported the research questions was coded to its central idea. Different statements might have the same code if they possessed the same idea. Each problem (1, 2, and 3) was coded in this study. The next step was to validate coded segments to establish the final initial code structure. To reach this, researchers conducted a discussion, such as merging codes with low frequency into one code. However, not all low-frequency codes could be combined since essential statements from the data are only mentioned once. From the careful and thorough code naming, 22 codes were determined: 5, 8, and 9 for Problem 1, Problem 2, and Problem 3, respectively.

(3) Grouping codes into categories

At this stage, conceptually similar codes were grouped into categories. The purpose of building categories was to help in sorting ideas, which was central to the characteristics of students' meaning in solving functional relationship problems. The researchers conducted a comprehensive discussion, and later, it was decided that there were nine categories established: 2 categories for Problem 1, 3 for Problem 2, and Problem 3.

(4) Integrating categories into themes

The following stage considered how the nine categories fit together and could be integrated to form themes. Looking back at the general research aim, to describe students' functional thinking when solving word problems, the researchers agreed to name five main themes to organize the nine categories: 1 for Problem 1, 2 for Problem 2, and 2 for Problem 3.

After ATLAS.ti had done the analysis process, confirmation from an external coder who was an expert in mathematics education and statistics was involved to ensure the objectivity of codes, categories, and themes obtained. Therefore, the joint data analysis from the students' written test and interview as well as the expert judgment for codes generated using ATLAS.ti guaranteed the validity and reliability of this study (triangulation of data).

Findings / Results

All students ($n = 39$) responded to identifying patterns in the first two problems. Meanwhile, seven students could not identify the pattern in the last problem. Table 1 summarizes the functional relationships evidenced by the student's responses to each problem from the written test and individual interview. Additionally, the themes, categories, and codes for students' responses presented in Table 1 were processed using the ATLAS.ti software.

As the data in Table 1, it was decided that each code assigned a unique label for convenience of explanation later. For instance, the code 1-C2 implies Problem Number 1 (1), Correspondence (C), Second Code (2), which is "*Generating the formula (9)*". The number after each code's definition refers to the reference sources derived from students' responses to the written test and interview transcripts. Hence, "*Generating the formula*" appears nine times in the responses.

Table 1. Themes, Categories, and Codes Result in Students' Responses to The Problem

Theme	Category	Code	Code's Definition
Problem Number 1			
Recognizing the functional relationship	Correspondence	1-C1	Finding the pattern with connection (9)
		1-C2	Generating the formula (9)
	Recursive patterns	1-R1	Finding the pattern without connection (32)
		1-R2	Considering n as the difference between two consecutive terms (15)
		1-R3	Considering n as the following number(s) (22)
Problem Number 2			
Recognizing the functional relationship	Correspondence	2-C1	Finding the pattern with connection (3)
		2-C2	Generating the formula incompletely (1)
		2-C3	Generating the formula completely (2)
	Recursive patterns	2-R1	Finding the pattern without connection (47)
		2-R2	Considering n as the difference between two consecutive terms (10)
		2-R3	Considering n as the following number(s) (21)
		2-R4	Generating the formula (focusing only on one variable) (8)
Unable to recognize the functional relationship	Difficulty	2-D1	Did not answer question (1)
Problem Number 3			
Recognizing the functional relationship	Correspondence	3-C1	Finding the pattern with connection (8)
		3-C2	Generating the formula incorrectly (2)
		3-C3	Generating the formula correctly (3)
	Recursive patterns	3-R1	Finding the pattern without connection (24)
		3-R2	Considering n as the difference between two consecutive terms (3)
		3-R3	Considering n as the following number (15)
		3-R4	Generating the formula (focusing only on one variable) (2)
Unable to recognize the functional relationship	Difficulty	3-D1	Unable to determine the values of the dependent variable (13)
		3-D2	Did not answer question (1)

Broadly speaking, the *difficulty* category is the least evidence provided by students, proceeded by the *correspondence* and the *recursive patterns* in all problems (Table 1). The students' responses indicate that most of them can identify patterns that are not informed explicitly in each problem. Moreover, recursive patterns seem to be the most popular way for students to generate patterns from specific values. This study predicted that learners would increasingly draw more attention to the recursive patterns from Problems 1, 2, and 3 (code 1-R1, 2-R1, 3-R1). However, the result reveals that recursively generating patterns are more prevalent in Problem 2, followed by Problem 3, then Problem 1. Typically, few students have succeeded in abstracting the generalization into formulas (code 1-C2 and 3-C3), whereas none obtain the correct formula for Problem 2 in the written test.

To provide examples of each identified functional relationship, the following section describes ways students generating patterns and creating formulas to symbolize the relationship between variables.

Functional Relationships Provided by Students in Identifying Patterns

In Problem 1, students were asked to fill in several values of the dependent variable, *the width of eroded soil,* based on the information displayed on the graph. The graph depicts the year (1, 2, and 3) and its associated soil erosion width (5 meters, 10 meters, and 15 meters). In this case, students were required to continue the width of eroded land in the fourth, fifth, and sixth years and determine the formula of the width of eroded soil in the nth year. The functional relation between variables employs the function $y = ax$.

There are 7 out of 39 students evidenced with correspondence in generating the pattern in this problem. Meanwhile, the rest provided the recursive patterns approach in their answer and neither generalized the relationships between variables. Following their written answer, we interviewed two students, one from correspondence and one from recursive patterns, to capture detailed information on how they determined each width of the eroded soil. Examples of students' interview transcripts are described in Table 2.

Table 2. Example of Students' Interview Transcripts in Generating the Pattern from Problem 1

Question	Student	Students' Answer
(1) How do you determine the eroded soil's width from the fourth to the sixth year? (2) What can you infer from the relationship between the year and the width of eroded soil?	S1	(1) I read from the graph that the width increases by 5 meters yearly. So, the fourth year is 4×5, the fifth year is 5 × 5, and so on. (2) the width of eroded soil is 5 times the year.
	S2	(1) I have to add 5 meters from the soil's width in the previous year. (2) It always increases by 5 meters every year.

According to their explanation (Table 2), S1 managed to find the eroded soil width from the first to the sixth year. S1 provided evidence that not only she did identify the pattern (5 meters each year), but she also recognized how the independent and dependent variables relate to each other (1-C1). Thus, she built a correspondence between the year (4, 5, and 6) and the width (20, 25, and 30). Nevertheless, instead of identifying how the change in the year related to the change in the width of eroded soil, S2's answer (Table 2) refers that she recursively added 5 meters from the previous width (1-R1).

Furthermore, the functional relation between "*The Day*" as the independent variable and "*Juna's total savings*" as the dependent variable in Problem 2 utilizes the function $y = ax + b$. According to the student's answers in the written test, three students' responses were categorized as correspondence in generating the pattern from Problem 2, and 36 others were evidenced for employing recursive patterns approach. The interview result from S3, the representative for correspondence, and S4, for recursive patterns, are transcribed in Table 3 below.

Table 3. Example of Students' Interview Transcripts in Generating the Pattern from Problem 2

Question	Student	Students' Answer
(1) How do you determine the total of Juna's savings from the fourth until the ninth day?	S3	(1) Juna's total savings on Day 4 is 4 times 4000, the money he saved every day, plus 30.000, the money he already owned. Thus, I used the same method for Day 5 until Day 9. (2) the total savings is 4000 times the day plus 30.000
(2) What can you infer from the relationship between the day and Juna's total savings?	S4	(1) Juna's total savings on Day 4 is the total savings from the previous day plus 4000 rupiahs. (2) The total savings increases by 4000 rupiahs every day

S3's response (Table 3) could correspond to finding specific values of the dependent variable (2-C1). She was able to determine how the change in Day (the independent variable) could relate to the change in Juna's total savings (The dependent variable) in local instances. For instance, *if the day is 4, the total savings is* $4 \times 4000 + 30000$. *If the day is 5, then the total savings is* $5 \times 4000 + 30000$. Notice that she identified the difference between two consecutive values (4000 rupiahs) and included the constant (30000 rupiahs) in determining each value of the dependent variable. However, in determining the values of the dependent variable, *Juna's savings*, S4 only attended to the *plus 4000 rupiahs* and recursively calculated the dependent variable's values. S4's answer implies that he only focused on the change of one quantity, namely the dependent variable (2-R1).

Furthermore, students' evidence for correspondence was also found in their answers to Problem 3 (5 out of 24). The problem context, *cake being sold* as the independent variable and *profit* as the dependent variable, required students to identify the pattern from the written information. The function rule applied in this problem is $y = ax - b$. Nevertheless, among those five students, only one student generated the formula correctly, one student miswrote the formula, and three students managed to make the formula during interviews. Moreover, responses from 20 students imply that they were recursively determining values of the dependent variable, i.e., by adding the former value with the same number. Table 4 shows the students' responses to correspondence and recursive patterns during their interviews.

Table 4. Example of Students' Interview Transcripts in Generating the Pattern from Problem 3

Question	Student	Students' Answer
(1) How do you determine the profit if one to six cakes are sold?	S5	(1) If Nia sells a cake, she gets profit 25.000 minus 20.000, the rental fee. Then, if Nia sells two cakes, the profit is 2 times 25.000 minus 20.000. I used the same method until six cakes. (2) The profit is the cakes times 25.000 minus 20.000.
(2) What can you infer from the relationship between the cakes being sold and the profit?	S6	(1) If Nia sold a cake, the profit is 25.000 minus 20.000, the rental fee. Then, if Nia sold two cakes, I must add 25.000 from the previous profit, and so on. (2) The profit increases by 25.000 for every cake being sold.

During the interview, S5 explained how he found the profit from one to six cakes sold (Table 4). It is noticed that S5 provided a correspondence way in relating two variables and not merely focused on the change in the dependent variable (3-C1). Nonetheless, the response from S6 (Table 4) indicates that he determined values of the dependent variable recursively, i.e., by adding the former value with the same number. S6's answer is the typical explanation uttered by students with the recursive patterns approach (3-R1).

Students' Ability to Symbolize the Generalization According to Their Functional Relationships Approach

In Problem 1, students that managed to see the functional relationship between corresponding pairs of variables (the year and the width of eroded soil) also managed to determine the formula for the width of eroded soil in the nth year. There are two types of formulas that the students wrote: $5n$ and $5 \times n$ (1A-C2). Meanwhile, students who recursively identified the functional relationship between variables could not write down the formula. Some students wrote the formula as $5m$, instead of $5n$. They struggled to understand the meaning of year n in the table and how they were supposed to calculate the width of eroded soil in the year n. Consequently, they assumed that the formula means *the pattern*, which is 5 meters (1A-R2). Similarly, the other students perceived year n as the seventh year (the previous one was the sixth year), so they determined the formula as $35\ m$ (the width of eroded soil in the seventh year) (1A-R3). Another student considered year n as the following years (7, 8, 9, 10, and so on), so she stated the formula as $35, 40, 45$, and so forth (the width of eroded soil in the year 7, 8, 9, 10, and so forth) (1A-R3). Figure 2 displays examples of students' answers in formulating the values of the dependent variable in Problem 1.

Year	The width of eroded soil
1	5 m
2	10 m
3	15 m
4	20 m
5	25 m
6	30 m
n	5 n

(a)

Year	The width of eroded soil
1	5
2	10
3	15
4	20
5	25
6	30
n	35,40,45 50,55..

(b)

Figure 2. Example of Students' Formulation in Problem 1: Correspondence (a) and Recursive Patterns (b)

Despite recognizing the pattern with correspondence in Problem 2, none of the students provides the correct formula for *Juna's total savings on Day n* in the written test. S3's answer implies that he just focused on the number since he only wrote $30000 + 4000$ (2-C2). Meanwhile, the other two succeeded in formulating $4000n + 30000$ during the interview section (2-C3). Moreover, students' responses to recursive patterns in Problem 2 were relatively similar to Problem 1. Students' reliance on recursively calculating the dependent variable's values became an issue in generating the formula. Instead, they wrote the formula as follows: (1) 4000, the difference between two consecutive terms (2-R2), (2) 70.000, the following term (2-R3), (3) 70.000, 74.000, 78.000, and so forth, the following consecutive terms (2-R3), and (4) $25.000n$, the multiplication between the difference and n (2-R4). Figure 3 depicts examples of students' answers in writing the formula from Problem 2.

The Day	Juna's Total Savings
4	46.000
5	50.000
6	54.000
7	58000
8	62.000
9	66.000
n	30.000 + 4.000

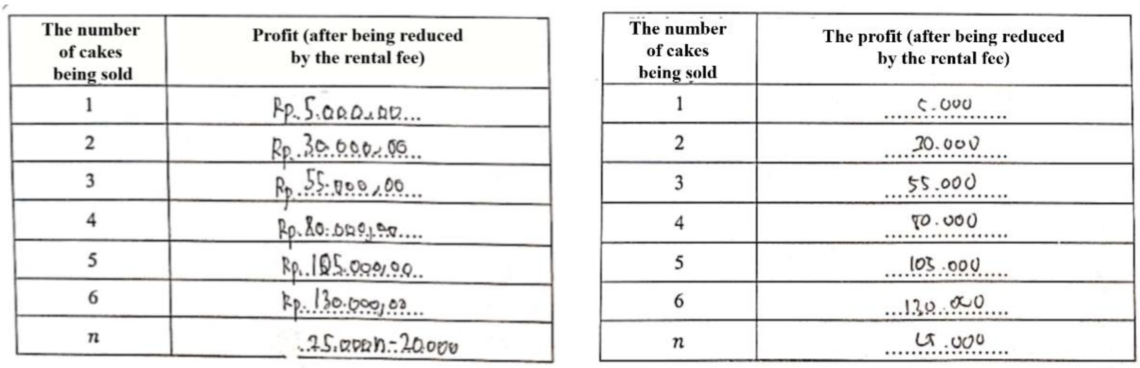
(a)

The Day	Juna's Total Savings
4	46.000
5	50.000
6	54.000
7	58.000
8	62.000
9	66.000
n	70,000

(b)

Figure 3. Example of Students' Formulation in Problem 2: Correspondence (a) and Recursive Patterns (b)

Furthermore, students that provided a correspondence way in relating two variables and not merely focused on the change in the dependent variable in Problem 3 managed to state the formula $25.000n - 20.000$ (3-C2). However, not all students with correspondence generated the formula correctly. For example, S7 struggled to symbolize the generalization as she wrote the incorrect formula, $25n - 20$ (3-C3). Akin to Problem 2, there were three types of formulas that students evidenced with recursive patterns produced: (1) 25.000, the difference between consecutive quantities (3-R2), (2) 155.000, the quantities in the next term (3-R3), and (3) $25.000n$, focused only to the difference in one variable (3-R4). Figure 4 illustrates how students create the formula for Problem 3.



The number of cakes being sold	Profit (after being reduced by the rental fee)
1	Rp. 5.000,00...
2	Rp. 30.000,00...
3	Rp. 55.000,00...
4	Rp. 80.000,00....
5	Rp. 105.000,00...
6	Rp. 130.000,00...
n	25.000n-20.000

(a)

The number of cakes being sold	The profit (after being reduced by the rental fee)
1	5.000
2	30.000
3	55.000
4	80.000
5	105.000
6	130.000
n	25.000

(b)

Figure 4. Example of Students' Formulation in Problem 3: Correspondence (a) and Recursive Patterns (b)

Discussion

This paper refers to the level of functional thinking ability generated by Doorman et al. (2012), Smith (2008), and Wilkie (2014) as the framework of the study, that is, *recursive patterns, covariation,* and *correspondence*. Moreover, the stages of students' generalization of patterns developed by Radford (2010) and Radford (2006) are used further to explain students' generalization in each functional thinking level. According to Radford and his peer, there are four stages of how students perform generalization from a given pattern: *arithmetic generalization, factual generalization, contextual generalization,* and *symbolic generalization*.

According to the result of this study, students who could identify the relationship between variables (independent and dependent) in general are categorized as having functional thinking ability at the correspondence level (Smith, 2008). The finding supports Doorman et al. (2012) that students with correspondence level can understand functions as a mathematical object which can be represented in multiple ways. In this study, the task given provides the function in natural language or graphical form, and each student must transform it to tabular until symbolic form. Addressing the stages of pattern generalization, typical students at the correspondence level could determine the function formula for any given problem ($5n, 30000n + 4000, 25000n - 20.000$). The stage, according to Radford (2010) and Radford (2006), belongs to the symbolic generalization, namely that students can express the generalization with alphanumeric symbols.

Furthermore, most students evidenced for correspondence in generating patterns perform no difficulties in abstracting the general value of the dependent variable. They described the relationship between two variables with a local rule so that a particular value of the independent variable could be used to calculate the corresponding value of the dependent variable (Wilkie & Ayalon, 2018). By identifying a local rule, they managed to generalize the rule so it could be applied to any value. The correspondence approach to functions is an essential aspect of school algebra (Usiskin, 1988) since it requires students to see a function as an object so that it can be manipulated into other representations (Lichti & Roth, 2018).

Nevertheless, not all students at the correspondence level could find the formula correctly. The result is similar to the study conducted by other researchers that few of them could correctly produce algebraic expressions of pattern generalizations (Wilkie, 2016). Moreover, this study also reveals that there were few students whose proficiency was at the correspondence level. Wilkie and Ayalon (2018) found that ninth graders tended to be able to write down formulas of a function more than any students in lower grades when it comes to geometric patterns. This study's finding complements their research that when the problem presented was in the form of word problems, only a few students with the same grade level could determine the function of that problem. The difference in students' ability to reach symbolic generalization in response to these two types of problems may be attributable to the findings of previous research. Generalizing geometric shapes was easier because students could observe the changing parts of objects in each sequence (Pinto & Cañadas, 2021; Radford, 2006; Ramírez et al., 2022; Wilkie, 2016; Wilkie & Clarke, 2016).

As opposed to students at the correspondence level, students who could only identify variations in a limited number of sequence patterns, according to Smith (2008), are categorized as having functional thinking ability at the recursive patterns level. Students at this level see functions as a request to calculate or a process of input-output assignment (Doorman et al., 2012; Lichti & Roth, 2018). It could be seen from the student's responses in the individual interview that when n was given as the independent variable, they calculated the dependent variable value and treated it as a local instead of general value. They summed up the n, representing the independent variable, as the following dependent variable's value. The result supports previous studies (Bajo-Benito et al., 2023; El Mouhayar, 2018; Wahyuni et al., 2020) that secondary graders commonly recognized patterns by finding the difference between two consecutive terms and then adding them to a quantity in a given to produce the quantity in the next term. In this case, student's conceptual understanding of variables is limited to the specific unknown and not to the "variability" as it applies to the function (Usiskin, 1988).

During the interview sections with selected students, this study implies that students with recursive patterns relied heavily on attending to variation within one quantity (Pinto & Cañadas, 2021). For instance, *the width increases by 5* (Problem 1), *just add 4000 rupiahs to the total savings* (Problem 2), and *the profit increases by 25.000 rupiahs* (Problem 3). Students in the recursive patterns level recognized local commonality among some quantities in the pattern without being able to provide an explicit generalization, known as an arithmetic generalization, or expressed the generality with natural language, known as a factual generalization (Radford, 2006, 2010). Indeed, the main difficulty for Indonesian students is transforming the problem into mathematical models, such as formulating expressions, within word problems (Jupri & Drijvers, 2016).

Furthermore, a table for each problem was intended to encourage students to see how the two variables are related (Confrey & Smith, 1995). In this case, they might demonstrate the covariation approach, i.e., how the change in one (the independent variable) produces a change in the other (the dependent variable) or the general relation between variables as in the correspondence (Confrey & Smith, 1994; Smith, 2008). Nevertheless, our findings provide evidences for Carraher and Schliemann's statement (Carraher & Schliemann, 2007) that tables of values did not necessarily make students realize the covariational changes in values nor the corresponding relation between variables since they relied on term-to-term changes in one variable. As a sequence, students could not represent the generality with symbolic expressions.

Overall, this study discovers a noticeable response from students' work that their functional thinking approach to solving the problem is highly affected by the type of function rule assigned in each problem. The more complex the function's rule, from $y = ax, y = ax + b$, to $y = ax - b$, students who were originally evidenced for the correspondence level slowly turned into the recursive patterns level. Likewise, students with the recursive patterns level were slowly unable to identify existing patterns. Therefore, the findings of this study supplement those of prior research (Blanton et al., 2017; Pinto et al., 2022; Stephens et al., 2017) that not only different types of functions affect the diversity of representations used by students but also the diversity of levels of students' functional thinking abilities.

Additionally, the functional thinking problems assigned to the students in this study align with the algebraic reasoning ability test developed by Basir et al. (2021). According to them, drawing conclusions about algebraic reasoning problems is through the processes of finding patterns, recognizing patterns, and generalizing the patterns found. By referring to the functional thinking levels to investigate further how students generalize patterns, this study found that students in different functional thinking levels have different ways of finding patterns and different views of understanding symbols.

Conclusion

This study provides evidences of how secondary graders (13-14 years old) focus on functional relationships when: (1) generating pattern from specific instances and (2) expressing the generality with a formula. Through the data analysis process using ATLAS.ti, this study found three main categories from students' responses in generating patterns: *correspondence, recursive patterns,* and *difficulties*.

In generating patterns, students with correspondence manage to build a relation between the corresponding pair of variables, then write the explicit generalization using a formula. The recursive patterns' students focus only on the change in one quantity (the dependent variable), so they are unable to produce a formula to represent the generalization. Many of them do not have adequate knowledge about variables. Additionally, students with difficulties cannot identify patterns.

Moreover, using word problems to explore students' functional thinking in this study contributes to two main findings in the existing studies. First of all, previous studies revealed that many nine-grade students manage to be at the correspondence level in generating geometric patterns; however, this study showed that few students reach the same level in generating patterns in word problems. Another finding is that different types of function rules, from easy to complex, affect not only students' representation but also their functional thinking levels. Students' functional thinking levels remain the same or degrade as the function rules become increasingly complex.

Recommendations

Functional thinking is essential for students learning about functions and for further calculus study. The ability entails students' proficiency in understanding the function as an object; functions exhibit equal values regardless of how functions are represented. In this case, it is recommended for mathematics secondary school teachers to provide learning activities that enhance students' functional thinking, especially encouraging students to be in the correspondence level. Further research might extend the investigation of students' functional thinking in building connections between functions' representations such as pairing sets, tabular, graphical, symbolic, and natural language.

Limitations

In this study, the analysis of functional relationships evidenced by students only includes positive functions within three functions' representation (graph, table, and symbols, in order). It did not conduct a more in-depth exploration of negative functions or how students' functional thinking generates functional relationships between variables using different representations interchangeably.

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Authorship Contribution Statement

Utami: Conceptualization, data acquisition, data analysis, drafting the manuscript. Prabawanto: Supervision, editing, and reviewing the manuscript, final approval. Suryadi: Supervision, editing, and reviewing the manuscript.

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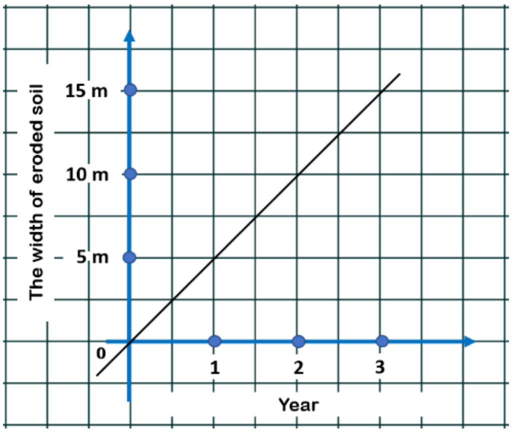
Appendix

Problem Context	Function
1. A house was built 100 meters away from the ravine's edge. As a result of soil erosion, the distance between the ravine and the house is constantly decreasing. The graph below shows the annual erosion of soil width.	$y = ax$



According to the graph above, fill in the blank in the following table.

Year	The width of eroded soil
1	……………………
2	……………………
3	……………………
4	……………………
5	……………………
6	……………………
n	……………………

Problem Context	Function
2. Juna plans to buy shoes by saving money. In order to buy these shoes, he saves his pocket money every day. Previously, in Juna's savings there was already 30.000 rupiah. On the first day he saved, the amount of money was 34.000 rupiah. On the second day, the amount of money was 38.000 rupiah. On the third day, the amount of money is 42.000 rupiah, and so on with the same amount of money saved every day. Fill in the blank in the following table.	$y = ax + b$

The Day	Juna's total savings
4	……………………
5	……………………
6	……………………
7	……………………
8	……………………
9	……………………
n	……………………

Problem Context	Function
3. A food bazaar was held for one day. Each seller who participates in the bazaar is charged a rental fee of 20.000 rupiah which is paid after the bazaar is ended. Nia rented a place to sell cakes for 25.000 rupiah/cake. Fill in the blank in the following table.	$y = ax - b$

The number of cakes being sold	The profit (after being reduced by the rental fee)
1	……………………
2	……………………
3	……………………
4	……………………
5	……………………
6	……………………
n	……………………